Self-dual models and mass generation in planar field theory

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We analyze, in three space-time dimensions, the connection between Abelian self-dual vector doublets and their counterparts containing both an explicit mass and a topological mass. Their correspondence is established in the Lagrangian formalism using an operator approach as well as a path integral approach. A canonical Hamiltonian analysis is presented, which also shows the equivalence with the Lagrangian formalism. The implications of our results for bosonization in three dimensions are discussed.

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I. INTRODUCTION

Self-dual models in three space time dimensions have certain distinct features which are essentially connected with the presence of the Chern-Simons term which is both metric and gauge independent. An important variant of such a model is the topologically massive gauge theory [1,2] where gauge invariance coexists with the finite mass, single helicity and parity violating nature of the excitations. Its dynamics is governed by a Lagrangian comprising both the Maxwell and Chern-Simons terms. The equations of motion, when expressed in terms of the dual to the field tensor, manifest a self-duality. An equivalent version of this model also exists, where the self-duality is revealed in the equations of motion for the basic field [3–5]. More recently, another possibility has been considered where, instead of the first derivative Chern-Simons term, a parity violating third derivative term is added to the Maxwell term [6].

An intriguing fact, first noted in [2] and briefly discussed in [7–9], is that topologically massive doublets, with identical mass parameters having opposite sign, are equivalent to a parity preserving vector theory with an explicit mass term. This is the Proca model. The invariance of the doublets under the combined parity and field interchanges is thereby easily understood from the equivalent theory. Moreover the two theories of the doublet characterize self- and anti-self-dual solutions, depending on the sign of the mass term. The final effective theory, which is a superposition of these solutions, therefore hides these symmetries.

In this paper we will make a detailed analysis of a doublet of topologically massive theories with distinct mass parameters. The resultant theory is a parity violating non-gauge vector theory with explicit as well as topological mass terms. This is demonstrated in Sec. II in the Lagrangian formalism using an operator approach. These results are then interpreted in the path integral approach. A Hamiltonian reduction of the effective theory, based on canonical transformations, is performed in Sec. III. The diagonalization of the Hamiltonian reveals the presence of two massive modes, which are a combination of topological and explicit mass parameters. These modes can be identified with those of the original Maxwell-Chern-Simons doublet thereby revealing a complete equivalence with the Lagrangian formalism. The diagonalization of the energy-momentum tensor is carried out in Sec. IV. Following a method elaborated in [2], the spin of the excitations is calculated. The helicity states are ± 1, corresponding to the two modes of the theory. An application to the bosonization of a doublet of massive Thirring models in the long wavelength limit is discussed in Sec. V. Our concluding remarks are left for Sec. VI.

II. LAGRANGIAN ANALYSIS

A. An operator approach

In this section we shall consider a doublet of self- and anti-self-dual models whose dynamics is governed, respectively, by the following Lagrangian densities:

$$\mathcal{L}_{SD}=\mathcal{L}_{-}=\frac{m_{-}}{2}g_{\mu}g^{\mu}-\frac{1}{2}\epsilon_{\mu\nu\lambda}g^{\mu}\partial^{\nu}g^{\lambda} \tag{2.1}$$

$$\mathcal{L}_{ASD}=\mathcal{L}_{+}=\frac{m_{+}}{2}f_{\mu}f^{\mu}+\frac{1}{2}\epsilon_{\mu\nu\lambda}f^{\mu}\partial^{\nu}f^{\lambda}. \tag{2.2}$$

The property of self- (or anti-self-)duality follows on exploiting the equations of motion [8]. Note that the mass parameters are different in the two cases. It has been suggested [9] that the above models combine to yield the Maxwell-Chern- Simons model with a conventional mass term. Here we quickly review that approach, which is based on [10]. The idea is to construct an effective Lagrangian that will characterize the doublet. Obviously a simple minded addition of the two Lagrangians will not yield anything. A new field will have to be introduced which will glue or solder the two Lagrangians. The final or effective Lagrangian will not contain this new field. Later on we shall show in what sense this approach can be understood as an ''addition'' of the two Lagrangians. Consider the variation of the Lagrangians under the local transformation

$$\delta f_{\mu}=\delta g_{\mu}=\Lambda_{\mu}(x). \tag{2.3}$$

The requisite variations are given by

$$\delta\mathcal{L}_{\mp}=J^{\mu}_{\mp}\Lambda_{\mu} \tag{2.4}$$

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where the currents are defined as

$$J^{\mu}_{\mp}=m_{\mp}h^{\mu}\mp\epsilon^{\mu\alpha\beta}\partial_{\alpha}h_{\beta};\quad h=f,g. \tag{2.5}$$

Next we introduce the soldering field W_{μ} transforming as

$$\delta W_{\mu}=\Lambda_{\mu}. \tag{2.6}$$

It is now simple to check that the following Lagrangian:

$$\mathcal{L}=\mathcal{L}_{-}(g)+\mathcal{L}_{+}(f)-W_{\mu}(J^{\mu}_{+}(f)+J^{\mu}_{-}(g))+\frac{1}{2}(m_{+}+m_{-})W_{\mu}W^{\mu} \tag{2.7}$$

is invariant under the transformations introduced earlier. The field W_{μ} plays the role of an auxiliary variable that can be eliminated by using the equation of motion,

$$W_{\mu}=\frac{1}{m_{+}+m_{-}}(J^{+}_{\mu}(f)+J^{-}_{\mu}(g)). \tag{2.8}$$

The final theory is manifestly invariant under the transformations containing only the difference of the original fields. It is given by

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}(A)+\frac{1}{2}\epsilon_{\mu\nu\lambda}(m_{-}-m_{+})A^{\mu}\partial^{\nu}A^{\lambda}+\frac{1}{2}m_{+}m_{-}A_{\mu}A^{\mu} \tag{2.9}$$

where

$$A_{\mu}=\frac{1}{\sqrt{m_{+}+m_{-}}}(f_{\mu}-g_{\mu}). \tag{2.10}$$

This is the Maxwell-Chern-Simons theory with an explicit mass term. A word about the degree of freedom count might be useful. The Lagrangians (2.1) and (2.2) individually correspond to single massive modes. The composite model (2.9) corresponds to two massive modes. There is thus a matching of the degree of freedom count.

It is now possible to take a different variation of the fields, but the final result will be the same. To illustrate this consider, instead of Eq. (2.3), the following variations:

$$\delta f_{\mu}=\delta g_{\mu}=\epsilon_{\mu\alpha\beta}\partial^{\alpha}\Lambda^{\beta}. \tag{2.11}$$

The variations in the individual Lagrangians can be written in terms of the parameter Λ as

$$\delta\mathcal{L}_{\mp}=J^{\alpha\beta}_{\mp}\partial_{\alpha}\Lambda_{\beta}, \tag{2.12}$$

where

$$J^{\alpha\beta}_{\mp}=m_{\mp}\epsilon^{\alpha\beta\mu}h_{\mu}\mp h^{\alpha\beta};\quad h=f,g \tag{2.13}$$

and

$$h^{\alpha\beta}=\partial^{\alpha}h^{\beta}-\partial^{\beta}h^{\alpha}. \tag{2.14}$$

Introducing an antisymmetric tensor field $B_{\alpha\beta}$ transforming as

$$\delta B_{\alpha\beta}=\partial_{\alpha}\Lambda_{\beta}-\partial_{\beta}\Lambda_{\alpha} \tag{2.15}$$

it is possible to write a modified Lagrangian,

$$\mathcal{L}=\mathcal{L}_{SD}+\mathcal{L}_{ASD}-\frac{1}{2}B^{\alpha\beta}(J^{+}_{\alpha\beta}(f)+J^{-}_{\alpha\beta}(g))+\frac{1}{4}(m_{+}+m_{-})B_{\alpha\beta}B^{\alpha\beta} \tag{2.16}$$

that is invariant under Eqs. (2.11) and (2.15); i.e., $\delta\mathcal{L}=0$. Since $B_{\alpha\beta}$ is an auxiliary field it is eliminated from Eq. (2.16) by using its solution. The final effective theory is just Eq. (2.9).

The above manipulations have shown that it is possible to glue the two Lagrangians by introducing an auxiliary variable. We could adopt this method to glue any two Lagrangians; however the final result would not be local. The local expression follows precisely because the self- and anti-self-dual nature of the Lagrangians engage in a canceling act. Note that the variations considered here lead to the combination $f_{\mu}-g_{\mu}$ in the effective theory. By considering the variations with opposite signatures we would have been led to the same effective theory but with the combination $f_{\mu}+g_{\mu}$.

As announced earlier we now show how the above approach enables one to directly obtain the effective theory by adding the two Lagrangians,

$$\mathcal{L}=\mathcal{L}_{+}(f)+\mathcal{L}_{-}(g). \tag{2.17}$$

Introducing the combination (2.10), we find

$$\begin{aligned}\mathcal{L}&=\mathcal{L}_{+}(\sqrt{m_{+}+m_{-}}A+g)+\mathcal{L}_{-}(g)\\&=\frac{m_{+}}{2}(m_{+}+m_{-})A^{\mu}A_{\mu}+\frac{1}{2}(m_{+}+m_{-})g^{\mu}g_{\mu}\\&\quad+\sqrt{m_{+}+m_{-}}\epsilon_{\mu\nu\lambda}g^{\mu}\partial^{\nu}A^{\lambda}+m_{+}\sqrt{m_{+}+m_{-}}A_{\mu}g^{\mu}\\&\quad+\frac{m_{+}+m_{-}}{2}\epsilon_{\mu\nu\lambda}A^{\mu}\partial^{\nu}A^{\lambda}.\end{aligned} \tag{2.18}$$

Now g_{μ} behaves as an auxiliary variable. It is eliminated in favor of the other variable by using the equation of motion. The end result reproduces Eq. (2.9).

The compatibility of the equations of motion of the doublet and the effective theory is next shown. From Eqs. (2.1) and (2.2) the following equations are obtained:

$$g_{\mu}=\frac{1}{m_{-}}\epsilon_{\mu\nu\lambda}\partial^{\nu}g^{\lambda} \tag{2.19}$$

$$\partial_{\beta}g^{\mu\beta}=m_{-}\epsilon^{\mu\alpha\beta}\partial_{\alpha}g_{\beta} \tag{2.20}$$

and

$$f_\mu=-\frac{1}{m_+}\epsilon_{\mu\nu\lambda}\partial^\nu f^\lambda \tag{2.21}$$

$$\partial_\beta f^{\mu\beta}=-m_+\epsilon^{\mu\alpha\beta}\partial_\alpha f_\beta . \tag{2.22}$$

Using the above sets of equations it follows that

$$-\partial^\nu(f_{\mu\nu}-g_{\mu\nu})+(m_--m_+)\epsilon_{\mu\nu\lambda}\partial^\nu(f^\lambda-g^\lambda)$$
$$+m_+m_-(f_\mu-g_\mu)=0 \tag{2.23}$$

which is just the equation of motion for the effective Lagrangian (2.9) with the identification (2.10).

Now the self-dual model is known to be equivalent to the Maxwell-Chern-Simons theory [4,5]. Consequently the above analysis can be repeated for a doublet of Maxwell-Chern-Simons theories defined by the Lagrangian densities,

$$\mathcal{L}_-(P)=-\frac{1}{4m_-}F_{\mu\nu}F^{\mu\nu}(P)+\frac{1}{2}\epsilon_{\mu\nu\lambda}P^\mu\partial^\nu P^\lambda \tag{2.24}$$

$$\mathcal{L}_+(Q)=-\frac{1}{4m_+}F_{\mu\nu}F^{\mu\nu}(Q)-\frac{1}{2}\epsilon_{\mu\nu\lambda}Q^\mu\partial^\nu Q^\lambda . \tag{2.25}$$

Specifically, the models (2.24) and (2.25) are the analogues of those given in Eqs. (2.1) and (2.2), respectively. For the sake of comparison, the mass parameters $m_\mp$ are taken to be identical in both cases.

Now consider the variations of the Lagrangians under the following transformations:

$$\delta P_\mu=\delta Q_\mu=\Lambda_\mu . \tag{2.26}$$

Then it follows

$$\delta\mathcal{L}_\mp=J^\mp_{\mu\nu}\partial^\mu\Lambda^\nu \tag{2.27}$$

where

$$J^\mp_{\mu\nu}(W)=-\frac{1}{m_\mp}F_{\mu\nu}(W)\pm\epsilon_{\mu\nu\lambda}W^\lambda;\quad W=P,Q. \tag{2.28}$$

Introducing the $B_{\mu\nu}$ field transforming as Eq. (2.15), it is seen that the following combination

$$\mathcal{L}=\mathcal{L}_-(P)+\mathcal{L}_+(Q)-\frac{1}{2}B_{\mu\nu}(J_+^{\mu\nu}+J_-^{\mu\nu})$$
$$-\frac{1}{4}\left(\frac{1}{m_+}+\frac{1}{m_-}\right)B_{\mu\nu}B^{\mu\nu} \tag{2.29}$$

is invariant under the relevant transformations (2.15) and (2.26).

As before, the auxiliary field $B_{\mu\nu}$ is eliminated from Eq. (2.29) to yield the Lagrangian (2.9) in terms of a composite field which is the difference of the fields in the doublet,

$$A_\mu=\frac{1}{\sqrt{m_++m_-}}(P_\mu-Q_\mu). \tag{2.30}$$

The other considerations discussed for the self-dual models are all applicable here.

B. Path integral derivation

The above discussion has a natural interpretation in the path integral formalism. The point is that the analysis related to Eqs. (2.17) and (2.18) shows that it is possible to obtain the effective theory by an addition of the Lagrangians and then identifying an auxiliary variable which is eventually eliminated. Since the problem is Gaussian it is straightforward to interpret it in the path integral language. The elimination of the auxiliary variable just corresponds to a Gaussian integration over that variable. Let us therefore consider the following generating functional[1] for the doublet of self- and anti-self-dual models (2.1) and (2.2),

$$\mathcal{Z}=\int df_\mu dg_\mu \exp\left(i\int d^3x\left[\mathcal{L}_-(g)+\mathcal{L}_+(f)\right.\right.$$
$$\left.\left.+\frac{1}{\sqrt{m_++m_-}}(f_\mu-g_\mu)J^\mu\right]\right) \tag{2.31}$$

where a source has been introduced that is coupled to the difference (2.10) of the variables. A relabeling of variables as in (2.10) is made for which the Jacobian is trivial. The path integral is now rewritten in terms of the redefined variable A_μ and g_μ,

$$\mathcal{Z}=\int dA_\mu dg_\mu \exp\left(i\int d^3x\left[\frac{m_+}{2}(\sqrt{m_++m_-}A_\mu+g_\mu)^2\right.\right.$$
$$+\frac{1}{2}\epsilon_{\mu\nu\lambda}(\sqrt{m_++m_-}A^\mu+g^\mu)\partial^\nu(\sqrt{m_++m_-}A^\lambda+g^\lambda)$$
$$\left.\left.+\frac{m_-}{2}g_\mu g^\mu-\frac{1}{2}\epsilon_{\mu\nu\lambda}g^\mu\partial^\nu g^\lambda+A_\mu J^\mu\right]\right). \tag{2.32}$$

Integrating over the g_μ variable yields

$$\mathcal{Z}=\int dA_\mu \exp\left(i\int d^3x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right.\right.$$
$$\left.\left.+\frac{1}{2}(m_--m_+)\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda+\frac{m_+m_-}{2}A_\mu A^\mu+A^\mu J_\mu\right]\right). \tag{2.33}$$

In the absence of sources this is just the partition function for the Maxwell-Chern-Simons-Proca model (2.9). Furthermore,

[1]Note that the path integral following from the Hamiltonian version [5] requires the factor $\delta[f_0+(1/m_+)\epsilon_{ij}\partial_i f_j]\delta[g_0-(1/m_-)\epsilon_{ij}\partial_i g_j]$ in the measure to account for the constraints. Since this is a Gaussian problem the result of the path integral remains unaltered even if these factors are not included. This is how we choose to define the basic lagrangian path integral for the self- and anti-self-dual models.

the A_μ field in Eq. (2.33) is related to the original doublet fields by exactly the same equation (2.10). This shows the equivalence of the results obtained by the two approaches.

It is equally possible to carry out a similar analysis for a doublet of Maxwell-Chern-Simons theories. However, a gauge fixing is necessary to account for the gauge invariance of these theories. As was shown in [5], through the use of master Lagrangians, the basic field in the self-dual model can be identified with the basic field in the Maxwell-Chern-Simons theory defined in the covariant gauge. We therefore consider the generating functional obtained from Eqs. (2.24),(2.25):

$$\mathcal{Z}=\int dP_\mu dQ_\mu \delta(\partial_\mu P^\mu)\delta(\partial_\mu Q^\mu) \times\exp\left(i\int d^3x\left[\mathcal{L}_-(P)+\mathcal{L}_+(Q)+\frac{1}{\sqrt{m_++m_-}}(P_\mu-Q_\mu)J^\mu\right]\right) \tag{2.34}$$

where, as before, a coupling with an external source has been done with the difference (2.30) of the variables. Because of the gauge invariance of the integrand, the source J_μ should be conserved.

To perform the path integration, a renaming of variables according to Eq. (2.30) is done for which the Jacobian is trivial. Then,

$$\begin{aligned}\mathcal{Z}=\int dA_\mu dQ_\mu \delta(\partial_\mu A^\mu)\delta(\partial_\mu Q^\mu) &\times\exp\Bigg(i\int d^3x\Bigg[-\frac{1}{4m_-}(m_++m_-)F_{\mu\nu}(A)F^{\mu\nu}(A)\\ &-\frac{1}{4}\left(\frac{1}{m_+}+\frac{1}{m_-}\right)F_{\mu\nu}(Q)F^{\mu\nu}(Q)\\ &-\frac{\sqrt{m_++m_-}}{2m_-}F_{\mu\nu}(A)F^{\mu\nu}(Q)+\sqrt{m_++m_-}\,\epsilon_{\mu\nu\lambda}Q^\mu\partial^\nu A^\lambda\\ &+\frac{1}{2}(m_++m_-)\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda+A_\mu J^\mu\Bigg]\Bigg).\end{aligned} \tag{2.35}$$

Performing the integral over the Q_μ variables yields

$$\begin{aligned}\mathcal{Z}=\int dA_\mu \delta(\partial_\mu A^\mu) &\times\exp\Bigg(i\int d^3x\Bigg[-\frac{1}{4}F_{\mu\nu}(A)F^{\mu\nu}(A)+\frac{1}{2}(m_+m_-)A_\mu A^\mu\\ &+\frac{1}{2}(m_--m_+)\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda+A_\mu J^\mu\Bigg]\Bigg).\end{aligned} \tag{2.36}$$

Express the delta function in the measure by an integral over a variable α:

$$\begin{aligned}\mathcal{Z}=\int dA_\mu d\alpha\,\exp\Bigg(i\int d^3x\Bigg[\alpha\partial_\mu A^\mu-\frac{1}{4}F_{\mu\nu}(A)F^{\mu\nu}(A)\\ +\frac{1}{2}(m_+m_-)A_\mu A^\mu+\frac{1}{2}(m_--m_+)\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda\\ +A_\mu J^\mu\Bigg]\Bigg).\end{aligned} \tag{2.37}$$

Introducing a Stückelberg transformed field $A_\mu\to A_\mu+(m_+m_-)^{-1}\partial_\mu\alpha$ and using the conservation of the source (i.e., $\partial_\mu J^\mu=0$) it follows that

$$\begin{aligned}\mathcal{Z}=\int dA_\mu\exp\Bigg(i\int d^3x\Bigg[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}(m_+m_-)A_\mu A^\mu\\ +\frac{1}{2}(m_--m_+)\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda+A_\mu J^\mu\Bigg]\Bigg)\end{aligned} \tag{2.38}$$

where the integral over α has been absorbed in the normalization.

As before, the generating functional for the Maxwell-Chern-Simons theory with an explicit mass term is obtained. The connection of the basic field A_μ with the original doublet,of course, remains the same as in Eq. (2.30).

III. HAMILTONIAN REDUCTION AND CANONICAL TRANSFORMATIONS

The results of the previous section were achieved in the Lagrangian formulation by combining the doublet to yield the composite model. A complementary viewpoint will now be presented in the Hamiltonian formulation. By solving the constraint, the Hamiltonian of the model is expressed in term of a reduced set of variables. Next, by means of a canonical transformation, the Hamiltonian gets decomposed into two distinct pieces, which correspond to the Hamiltonians of the Maxwell-Chern-Simons doublet. This technique of using canonical transformations to diagonalize a Hamiltonian is of course well known and appears in different versions and different situations. More recently, in the context of the Lagrangian formalism discussed in Sec. II A, it has been developed in [11]. Defining a new set of parameters,

$$m_--m_+=\theta$$

$$m_+m_-=m^2 \tag{3.1}$$

the Lagrangian (2.9) takes the form

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{\theta}{2}\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda+\frac{m^2}{2}A_\mu A^\mu. \tag{3.2}$$

The canonical momenta are

$$\pi_i=\frac{\partial\mathcal{L}}{\partial\dot{A}^i}=-\left(F_{0i}+\frac{\theta}{2}\epsilon_{ij}A_j\right) \tag{3.3}$$

while

$$\pi_0 \approx 0 \tag{3.4}$$

is the primary constraint. The canonical Hamiltonian is given by

$$H=\frac{1}{2}\int d^2x\left[\pi_i^2+\frac{1}{2}F_{ij}^2+\left(\frac{\theta^2}{4}+m^2\right)A_i^2-\theta\epsilon_{ij}A_i\pi_j+m^2A_0^2\right]+\int d^2xA_0\Omega \tag{3.5}$$

where

$$\Omega=\partial_i\pi_i-\frac{\theta}{2}\epsilon_{ij}\partial_iA_j-m^2A_0\approx 0 \tag{3.6}$$

is the secondary constraint. Eliminating the multiplier A_0 from Eq. (3.5) by solving the constraint (3.6) we obtain

$$H=\frac{1}{2}\int d^2x\left[\pi_i^2+\left(\frac{1}{2}+\frac{\theta^2}{8m^2}\right)F_{ij}^2+\left(\frac{\theta^2}{4}+m^2\right)A_i^2-\theta\epsilon_{ij}A_i\pi_j\right]+\frac{1}{2m^2}\int d^2x[(\partial_i\pi_i)^2-\theta\partial_i\pi_i\epsilon_{lm}\partial_lA_m]. \tag{3.7}$$

Making the canonical transformations in terms of the new canonical pairs (α,π_α) and (β,π_β),

$$A_i=\frac{2m}{\sqrt{4m^2+\theta^2}}\epsilon_{ij}\frac{\partial_j}{\sqrt{-\partial^2}}(\alpha+\beta)+\frac{1}{2m}\frac{\partial_i}{\sqrt{-\partial^2}}(\pi_\alpha-\pi_\beta)$$

$$\pi_i=-\frac{\sqrt{4m^2+\theta^2}}{4m}\epsilon_{ij}\frac{\partial_j}{\sqrt{-\partial^2}}(\pi_\alpha+\pi_\beta)+m\frac{\partial_i}{\sqrt{-\partial^2}}(\alpha-\beta) \tag{3.8}$$

the Hamiltonian decouples into two independent pieces,

$$H(A_i,\pi_i)=H(\alpha,\pi_\alpha)+H(\beta,\pi_\beta) \tag{3.9}$$

where

$$H(\alpha,\pi_\alpha)=\frac{1}{16m^2}\sqrt{4m^2+\theta^2}(\sqrt{4m^2+\theta^2}-\theta)\int d^2x\pi_\alpha^2+\frac{\sqrt{4m^2+\theta^2}+\theta}{\sqrt{4m^2+\theta^2}}\int d^2x(\partial_i\alpha)^2+m^2\frac{(\sqrt{4m^2+\theta^2}-\theta)}{\sqrt{4m^2+\theta^2}}\int d^2x\alpha^2$$

$$H(\beta,\pi_\beta)=\frac{1}{16m^2}\sqrt{4m^2+\theta^2}(\sqrt{4m^2+\theta^2}+\theta)\int d^2x\pi_\beta^2+\frac{\sqrt{4m^2+\theta^2}-\theta}{\sqrt{4m^2+\theta^2}}\int d^2x(\partial_i\beta)^2+m^2\frac{(\sqrt{4m^2+\theta^2}+\theta)}{\sqrt{4m^2+\theta^2}}\int d^2x\beta^2. \tag{3.10}$$

To recast these expressions in a familiar form, a trivial scaling is done,

$$\alpha^2\to\frac{1}{2}\frac{\sqrt{4m^2+\theta^2}}{\sqrt{4m^2+\theta^2}+\theta}\alpha^2,\quad \pi_\alpha^2\to 2\frac{\sqrt{4m^2+\theta^2}+\theta}{\sqrt{4m^2+\theta^2}}\pi_\alpha^2$$

$$\beta^2\to\frac{1}{2}\frac{\sqrt{4m^2+\theta^2}}{\sqrt{4m^2+\theta^2}-\theta}\beta^2,\quad \pi_\beta^2\to 2\frac{\sqrt{4m^2+\theta^2}-\theta}{\sqrt{4m^2+\theta^2}}\pi_\beta^2 \tag{3.11}$$

so that

$$H(\alpha,\pi_\alpha)=\frac{1}{2}\int d^2x[(\partial_i\alpha)^2+\pi_\alpha^2+m_+^2\alpha^2]$$

$$H(\beta,\pi_\beta)=\frac{1}{2}\int d^2x[(\partial_i\beta)^2+\pi_\beta^2+m_-^2\beta^2] \tag{3.12}$$

with

$$m_\pm=\sqrt{m^2+\frac{\theta^2}{4}}\mp\frac{\theta}{2}. \tag{3.13}$$

These relations show that the theory possesses two massive modes with mass m_+ and m_- which satisfy the Klein Gordon equation. Furthermore since $m_\pm$ in Eq. (3.13) are the solutions to the set (3.1), these can be identified with the corresponding mass parameters occurring in the Maxwell-Chern-Simons doublet (2.24) and (2.25). The above Hamiltonians are indeed the reduced expressions obtained from Eqs. (2.25) and (2.24), respectively. The canonical reduction of the Maxwell-Chern-Simons theory has been done in [2] but we present it here from our viewpoint for the sake of completeness. Let us, for instance, consider the Lagrangian (2.24).[2] The multiplier A_0 enforces the Gauss constraint,

$$\Omega=\partial_i\pi_i-\frac{m_-}{2}\epsilon_{ij}\partial_iA_j\approx 0 \tag{3.14}$$

where (A_i,π^i) is a canonical set. The Hamiltonian on the constraint surface is given by

$$H=\frac{1}{2}\int d^2x\left[\pi_i^2+\frac{1}{2}F_{ij}^2+m_-\epsilon_{ij}\pi_iA_j+\frac{m_-^2}{4}A_i^2\right]. \tag{3.15}$$

[2]The variable P, for convenience, is now called A.

Next, consider the canonical transformation,

$$A_i=\frac{\partial_i}{\sqrt{-\partial^2}}\pi_\theta+\epsilon_{ij}\frac{\partial_j}{\sqrt{-\partial^2}}\beta$$

$$\pi_i=\frac{\partial_i}{\sqrt{-\partial^2}}\theta-\epsilon_{ij}\frac{\partial_j}{\sqrt{-\partial^2}}\pi_\beta \tag{3.16}$$

where (θ,π_θ) and (β,π_β) form independent canonical pairs. Since this is a gauge theory, a gauge fixing is imposed. We take the standard Coulomb gauge,

$$\partial_i A_i=0 \tag{3.17}$$

The presence of the gauge, together with the constraint (3.14), modifies the canonical structure of the (A_i,π_i) fields; i.e. their brackets are no longer canonical. The modified algebra can be obtained either by the Dirac algorithm [12] or, as done here, by just solving the constraints. Their solution leads to the following structure:

$$A_i=\epsilon_{ij}\frac{\partial_j}{\sqrt{-\partial^2}}\beta$$

$$\pi_i=-\frac{m_-}{2}\frac{\partial_i}{\sqrt{-\partial^2}}\beta-\epsilon_{ij}\frac{\partial_j}{\sqrt{-\partial^2}}\pi_\beta \tag{3.18}$$

which satisfies a nontrivial algebra,

$$[A_i(x),\pi_j(y)]=i\left(-\delta_{ij}+\frac{\partial_i\partial_j}{\partial^2}\right)\delta(x-y)$$

$$[\pi_i(x),\pi_j(y)]=-i\frac{m_-}{2}\epsilon_{ij}\delta(x-y). \tag{3.19}$$

The same result follows by replacing the Poisson bracket by the Dirac bracket. Using Eq. (3.18) the reduced Hamiltonian is obtained from Eq. (3.15),

$$H=\frac{1}{2}\int d^2x[(\partial_i\beta)^2+\pi_\beta^2+m_-^2\beta^2] \tag{3.20}$$

which has exactly the same structure as the second relation in Eq. (3.12). Likewise the other Maxwell-Chern-Simons theory with a coupling m_+ can be reduced to the first relation in Eq. (3.12). It might be mentioned that the two Lagrangians (2.24) and (2.25) differ not only in the respective mass parameters, but also in the signature of the Chern-Simons term. However a scaling argument shows that, apart from the field dependencies, these are connected by $m_+\to -m_-$. Since the Hamiltonian is quadratic in the mass term, this sign difference therefore does not affect the result.

Thus the reduced Hamiltonian of the Maxwell-Chern-Simons theory with a mass term is the sum of the reduced Hamiltonians of a doublet of Maxwell-Chern-Simons theories with distinct mass parameters $m_\pm$. There is a complete correspondence between the Lagrangian and Hamiltonian formulations.

IV. THE ENERGY MOMENTUM TENSOR AND SPIN

As emphasized in [2], spin in $2+1$ dimensions cannot be properly identified from only the angular momentum operator since it does not conform to the conventional algebra. It is essential to consider the complete energy momentum tensor. Incidentally, although α and β in Eq. (3.12) satisfy the Klein-Gordon equation, these cannot be regarded as scalars due to presence of the factor $\sqrt{-\partial^2}$ in the transformations (3.8). A complete analysis of the energy momentum tensor will be done which unambiguously determines the spin of the excitations. The energy momentum tensor following from Eq. (3.2) is given by

$$\begin{aligned}\Theta_{\mu\nu}&=2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}-g_{\mu\nu}\mathcal{L}\\&=-F_{\mu\alpha}F^\alpha_\nu+m^2A_\mu A_\nu\\&\quad+\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}-\frac{m^2}{2}g_{\mu\nu}A_\alpha A^\alpha.\end{aligned} \tag{4.1}$$

The discussion of the Hamiltonian has already been done. The momentum is given by

$$P_i=\int d^2x\Theta_{0i}=\int d^2x(-F_{0j}F^j_i+m^2A_0A_i). \tag{4.2}$$

To pass over to the reduced variables, A_0 is first eliminated by using the constraint (3.6). Next, the canonical transformations (3.8) and (3.11) are applied. This leads to the diagonal form,

$$P_i=\int d^2x[\pi_\alpha\partial_i\alpha+\pi_\beta\partial_i\beta]. \tag{4.3}$$

The rotation generator is given by

$$M_{ij}=\int d^2x[x_i\Theta_{0j}-x_j\Theta_{0i}] \tag{4.4}$$

which, following the same techniques, is put in the diagonal form,

$$\begin{aligned}M_{ij}=\int d^2x[&(x_i\pi_\alpha\partial_j\alpha-x_j\pi_\alpha\partial_i\alpha)\\&+(x_i\pi_\beta\partial_j\beta-x_j\pi_\beta\partial_i\beta)].\end{aligned} \tag{4.5}$$

Both the translation and rotation generators have their expected forms with the fields α and β transforming normally. Using the inverse transformation of Eq. (3.8) it is seen that the original field A_i also transforms normally,

$$[A_j,P_i]=i\partial_iA_j$$

$$[A_k,M_{ij}]=i(x_i\partial_j A_k-x_j\partial_i A_k-\delta_{ik}A_j+\delta_{jk}A_i). \tag{4.6}$$

Finally, the boosts are considered and it is found that the diagonal form is given by

$$\begin{aligned}M_{0i}&=t\int d^2x\Theta_{0i}-\int d^2x x_i\Theta_{00}\\&=t\int d^2x\pi_\alpha\partial_i\alpha-\frac{1}{2}\int d^2x x_i[(\partial_j\alpha)^2+\pi_\alpha^2+m_+^2\alpha^2]\\&\quad+m_+\epsilon_{ij}\int d^2x\pi_\alpha\left(\frac{\partial_j}{\partial^2}\right)\alpha+t\int d^2x\pi_\beta\partial_i\beta\\&\quad-\frac{1}{2}\int d^2x x_i[(\partial_j\beta)^2+\pi_\beta^2+m_-^2\beta^2]\\&\quad-m_-\epsilon_{ij}\int d^2x\pi_\beta\left(\frac{\partial_j}{\partial^2}\right)\beta.\end{aligned} \tag{4.7}$$

The boost generator has extra factors which clearly show that α and β do not transform as scalars. These extra pieces are however essential to correctly reproduce the usual transformation of the original vector field A_i,

$$[A_j,M_{0i}]=i(t\partial_i A_j-x_i\partial_0 A_j+\delta_{ij}A_0) \tag{4.8}$$

where recourse has to be taken to the solution of the constraint (3.6) to obtain the final structure involving A_0.

The presence of the abnormal terms in the boost leads to a zero momentum anomaly in the Poincaré algebra,

$$[M_{0i},M_{0j}]=i(M_{ij}+\epsilon_{ij}\Delta) \tag{4.9}$$

where

$$\begin{aligned}\Delta=&\frac{m_+^3}{4\pi}\left(\int d^2x\alpha\right)^2+\frac{m_+}{4\pi}\left(\int d^2x\pi_\alpha\right)^2-\frac{m_-^3}{4\pi}\left(\int d^2x\beta\right)^2\\&-\frac{m_-}{4\pi}\left(\int d^2x\pi_\beta\right)^2.\end{aligned} \tag{4.10}$$

Following exactly the same steps as in [2] it is possible to remove this anomaly, simultaneously fixing the spin of the excitations. Consider the mode expansions,

$$\begin{aligned}\alpha(x)&=\int\frac{d^2k}{2\pi\sqrt{2\omega(k)}}[a(k)e^{-ik\cdot x}+a^\dagger(k)e^{ik\cdot x}]\\\beta(x)&=\int\frac{d^2k}{2\pi\sqrt{2\omega(k)}}[b(k)e^{-ik\cdot x}+b^\dagger(k)e^{ik\cdot x}]\end{aligned} \tag{4.11}$$

suitably modified by the phase redefinitions,

$$a\rightarrow e^{-i\phi}a,\quad b\rightarrow e^{i\phi}b \tag{4.12}$$

where

$$\phi=\tan^{-1}\left(\frac{k_2}{k_1}\right). \tag{4.13}$$

It leads to the following expressions for the boosts and rotation generator:

$$\begin{aligned}M_{0i}=&\frac{i}{2}\int d^2k\omega(k)|a^\dagger(k)\overleftrightarrow{\partial}_i a(k)|\\&+\epsilon_{ij}\int d^2k\frac{1}{\omega(k)+m_+}k_j a^\dagger(k)a(k)\\&+\frac{i}{2}\int d^2k\omega(k)|b^\dagger(k)\overleftrightarrow{\partial}_i b(k)|\\&-\epsilon_{ij}\int d^2k\frac{1}{\omega(k)+m_-}k_j b^\dagger(k)b(k)\end{aligned} \tag{4.14}$$

$$\begin{aligned}M_{ij}=&\epsilon_{ij}\left(\int d^2k a^\dagger(k)\frac{1}{i}\frac{\partial}{\partial\phi}a(k)\right.\\&\left.-\int d^2k a^\dagger(k)a(k)\right)\\&+\epsilon_{ij}\left(\int d^2k b^\dagger(k)\frac{1}{i}\frac{\partial}{\partial\phi}b(k)\right.\\&\left.+\int d^2k b^\dagger(k)b(k)\right)\end{aligned} \tag{4.15}$$

which satisfy the Poincaré algebra

$$[M_{0i},M_{0j}]=iM_{ij}. \tag{4.16}$$

An inspection of the rotation generator shows that it comprises of two distinct terms denoted by the parentheses. The first factor in each corresponds to the usual orbital part. The additional pieces show that the spin of the excitations associated with α and β are, respectively, -1 and $+1$. This also happens in the case of the Maxwell-Chern-Simons theory [2]. The difference from the spin of the excitations in the Maxwell-Chern-Simons theory is noteworthy. There the sign of the spin is fixed by the sign of the coefficient of the Chern-Simons parameter. In the present case it is seen from Eq. (3.13) that, irrespective of the sign of θ, the mass parameters $m_\pm$ are always positive. Hence the sign of the spin associated with α and β is also uniquely determined.

Note that for $m_+=m_-$, the theory becomes parity conserving. This is the case when the Maxwell-Chern-Simons doublet with identical mass yields the Proca model [7,8].

V. APPLICATION TO 3D BOSONIZATION

Bosonization in higher dimensions is neither complete nor exact as in the case of two space-time dimensions. This is related to the fact that the fermion determinant in dimensions greater than two cannot be exactly computed. In general it has a nonlocal structure. However, for the large fermion mass limit in three space-time dimensions, a local expression emerges [2,13]. This has been exploited to discuss the bosonization of massive fermionic models in the long wavelength limit [14]. Here we analyze the bosonization of a doublet of such models: To be specific, consider the following three dimensional massive Thirring models:

$$\mathcal{L}_+ = \bar{\psi}(i\partial\!\!\!/ + m_+)\psi - \frac{\lambda_+^2}{2}(\bar{\psi}\gamma_\mu\psi)^2$$

$$\mathcal{L}_- = \bar{\chi}(i\partial\!\!\!/ - m_-)\chi - \frac{\lambda_-^2}{2}(\bar{\chi}\gamma_\mu\chi)^2. \tag{5.1}$$

The respective partition functions, after eliminating the four fermion interaction by introducing auxiliary fields, are given by

$$\mathcal{Z}_+ = \int d\psi d\bar{\psi} df_\mu \exp\left(i\int d^3x\left(\bar{\psi}(i\partial\!\!\!/ + m_+ + \lambda_+ f\!\!\!/)\psi + \frac{1}{2}f_\mu f^\mu\right)\right)$$

$$\mathcal{Z}_- = \int d\chi d\bar{\chi} dg_\mu \exp\left(i\int d^3x\left(\bar{\chi}(i\partial\!\!\!/ - m_- + \lambda_- g\!\!\!/)\chi + \frac{1}{2}g_\mu g^\mu\right)\right). \tag{5.2}$$

The fermion determinant can be expressed, in the large mass limit, by a local series involving (∂/m) [2,13,15]. Furthermore, for weak coupling we need to consider only the two legs fermion loop. The leading long wavelength term in this quadratic approximation is the Chern-Simons three form. Thus the effective bosonized Lagrangians of the doublet are given by

$$\mathcal{L}_+ = \frac{\lambda_+^2}{8\pi}\epsilon_{\mu\nu\lambda}f^\mu\partial^\nu f^\lambda + \frac{1}{2}f_\mu f^\mu + O\left(\frac{1}{m}\right)$$

$$\mathcal{L}_- = -\frac{\lambda_-^2}{8\pi}\epsilon_{\mu\nu\lambda}g^\mu\partial^\nu g^\lambda + \frac{1}{2}g_\mu g^\mu + O\left(\frac{1}{m}\right) \tag{5.3}$$

where the difference in the sign of the Chern-Simons piece is a result of a similar feature in the mass terms of the original Lagrangians (5.1).

Using our previous results, the doublet of $\mathcal{L}_+$ and $\mathcal{L}_-$, as defined in Eq. (5.3), can be represented by an effective Lagrangian, which is just the Maxwell-Chern-Simons theory with an explicit mass term,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}(A) + \frac{2\pi}{\lambda_+^2\lambda_-^2}(\lambda_+^2 - \lambda_-{}^{-2})\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda + \frac{8\pi^2}{\lambda_+^2\lambda_-^2}A_\mu A^\mu \tag{5.4}$$

where

$$A_\mu = \frac{\lambda_+\lambda_-}{\sqrt{4\pi(\lambda_+^2 + \lambda_-^2)}}(f_\mu - g_\mu). \tag{5.5}$$

The Lagrangian (5.4) can be regarded as a bosonized Lagrangian obtained from the following massive Thirring model:

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi - \frac{\lambda^2}{2}(\bar{\Psi}\gamma_\mu\Psi)^2 \tag{5.6}$$

in the weak coupling and large mass limit. The relations of the parameters occurring in the above Lagrangian and Eq. (5.4) are given by

$$m = \frac{8\pi}{3}\frac{(\lambda_+^2 - \lambda_-^2)}{\lambda_+^2\lambda_-^2}$$

$$\lambda^2 = \lambda_-^2 - \lambda_+^2. \tag{5.7}$$

To show this we first observe that the original weak coupling involving λ_+ and λ_- leads to a weak λ. Secondly, it also implies the large mass limit. In other words the same approximation prevails. The fermion determinant, similar to Eq. (5.3), but evaluated to the next to leading order, includes both the Chern-Simons term and the Maxwell term [15]. Specifically, this is written as

$$\mathcal{L} = \frac{1}{2}A^\mu A_\mu - \frac{\lambda^2}{8\pi}\epsilon\mu\nu\lambda A^\mu\partial^\nu A^\lambda + \frac{\lambda^2}{24\pi m}F^{\mu\nu}F_{\mu\nu} + O\left(\frac{1}{m^2}\right) \tag{5.8}$$

where we have identified the auxiliary field necessary to simplify the four fermion interaction with A_μ. This is exactly in keeping with the spirit of obtaining Eq. (5.3) from Eq. (5.2), except that the fermion determinant has been evaluated to the next to leading order in the inverse mass expansion. By making the following scaling:

$$A_\mu \to \frac{4\pi}{\lambda_+\lambda_-}A_\mu \tag{5.9}$$

this reproduces Eq. (5.4), with the identification (5.7). This establishes the connection between Eqs. (5.1) and (5.6) since Eq. (5.4) is their common origin.

The implications of the above analysis are now discussed. In the quadratic approximation, a doublet of massive Thirring models in the leading long wavelength limit bosonizes to the effective Lagrangian (5.4). The same effective theory, under similar approximations and with the identification (5.7), also characterizes a single massive Thirring model, but where the calculation of the fermion determinant is carried out till the first nonleading term. In this sense, therefore, a doublet of massive Thirring models can be approximated by a single similar model. There is also a matching in the degree of freedom count.

VI. CONCLUSIONS

We have considered the description of a doublet of self-dual models with distinct topological mass parameters having opposite signs. The difference in sign implies that the doublet comprises a self-dual and an anti-self-dual model. Specifically, this was a pair of the gauge invariant Maxwell-Chern-Simons theory [2] or, equivalently, its dual gauge variant version [3–5]. The effective theory, characterizing such a doublet, turned out to be the Maxwell-Chern-Simons theory with an explicit mass term. The basic field of the effective theory was just the difference of the doublet variables.

A canonical analysis of the effective theory was done. Based on a set of canonical transformations, the Hamiltonian was diagonalized into two separate pieces. The two massive modes were found to be a combination of the topological and explicit mass parameters. In fact these were identified with the two modes of the Maxwell-Chern-Simons doublet that led to the effective theory. In this way a correspondence was established between the Lagrangian approach of combining the doublet into an effective theory and the Hamiltonian approach of decomposing the latter back into the doublet. The spin of the excitations was obtained from a complete study of the Poincaré algebra by adopting the method advocated in [2].

When the Maxwell-Chern-Simons doublet has identical topological mass $\pm m$, parity is conserved since one degree of freedom is just mapped to the other. The spin carried by the two degrees of freedom is ∓ 1. This has the same kinematical structure as the Proca theory which is a parity conserving theory with two massive modes having spin ∓ 1 [16,2]. An explicit demonstration of this was provided earlier [7,8]. This result is reproduced here by putting $m_+ = m_-$.

For the more general case where the Maxwell-Chern-Simons doublet has different topological masses $m_\pm$, parity is no longer conserved, although the other considerations remain valid. Hence the kinematics of such a doublet resembles a non gauge parity violating theory with two massive modes having spin ∓ 1. This turned out to be the Maxwell-Chern-Simons theory with an explicit mass term, as elaborated here in details. An added bonus of this equivalence is that it led to fresh insights into the bosonization of massive fermionic models. This was explicitly shown for a doublet of massive Thirring models, but it can be done for other examples like QED in three dimensions.

Recently there have been certain discussions [17,18] which regard a mass term in a gauge theory either as a conventional mass term or, equivalently, as a gauge fixing term. In fact, Maxwell theory in the covariant gauge and the Proca model were shown equivalent from the viewpoint of quantum Becchi-Rouet-Stora-Tyutin (BRST) symmetry [17,18]. Here we find that the superposition of a pair of Maxwell-Chern-Simons theories in the covariant gauge leads to an explicit mass generation. This suggests a possible connection between these different approaches.

The extension of these findings to higher dimensions or non-Abelian versions would be welcome. Of course for $4k-1$ dimensions where self-duality is definable, this extension is straightforward in the Abelian case. For non-Abelian theories, the superposition principle does not work as in the Abelian theory. Using some special properties of two dimensions, the Wess-Zumino-Witten (WZW) non-Abelian doublet was treated in [19]. But for general dimensions, the non-Abelian analysis remains an open issue.

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